MODULE 2

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

18.3 PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING

A number of problems in engineering give rise to the following well-known partial differential equations :

(i) *Wave equation* : $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$.

(ii) *One dimensional heat flow equation* : $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

(iii) *Two dimensional heat flow equation* which in steady state becomes the two dimensional *Laplace's equation* : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(iv) *Transmission line equations.*

(v) *Vibrating membrane.* Two dimensional wave equation.

(vi) *Laplace's equation* in three dimensions.

Besides these, the partial differential equations frequently occur in the theory of Elasticity and Hydraulics.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solution gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

18.4 VIBRATIONS OF A STRETCHED STRING—WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T (Fig. 18.1). The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

Fig. 18.1

Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis ; so that the motion takes place entirely in the xy-plane. Figure 18.1 shows the string in the position APB at time t. Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with the x-axis. Clearly the element is moving upwards with the acceleration $\partial^2 y/\partial t^2$. Also the vertical component of the force acting on this element.
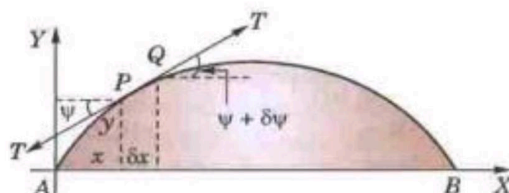
$$= T \sin(\psi + \delta\psi) - T \sin\psi = T[\sin(\psi + \delta\psi) - \sin\psi]$$

$$= T[\tan(\psi + \delta\psi) - \tan\psi], \text{ since } \psi \text{ is small} = T\left[\left\{\dfrac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\dfrac{\partial y}{\partial x}\right\}_x\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x \cdot \dfrac{\partial^2 y}{\partial t^2} = T\left[\left\{\dfrac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\dfrac{\partial y}{\partial x}\right\}_x\right] \quad i.e., \quad \dfrac{\partial^2 y}{\partial t^2} = \dfrac{T}{m}\left[\dfrac{\left\{\dfrac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\dfrac{\partial y}{\partial x}\right\}_x}{\delta x}\right]$$

Taking limits as $Q \to P$ *i.e.*, $dx \to 0$, we have $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$, where $c^2 = \dfrac{T}{m}$ ...(1)

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation*.

(2) Solution of the wave equation. Assume that a solution of (1) is of the form

$z = X(x)T(t)$ where X is a function of x and T is a function of t only.

Then $\frac{\partial^2 y}{\partial t^2} = X \cdot T''$ and $\frac{\partial^2 y}{\partial x^2} = X'' \cdot T$

Substituting these in (1), we get $XT'' = c^2 X''T$ *i.e.,* $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T}$ ...(2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2 X}{dx^2} - kX = 0 \quad ...(3) \quad \text{and} \quad \frac{d^2 T}{dt^2} - kc^2 T = 0 \quad\quad ...(4)$$

Solving (3) and (4), we get

(i) *When* k *is positive and* $= p^2$, *say* $X = c_1 e^{px} + c_2 e^{-px}$; $T = c_3 e^{cpt} + c_4 e^{-cpt}$.

(ii) *When* k *is negative and* $= -p^2$ *say* $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(iii) *When* k *is zero.* $X = c_9 x + c_{10}$; $T = c_{11}t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}) \quad\quad ...(5)$$
$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad\quad ...(6)$$
$$y = (c_9 x + c_{10})(c_{11}t + c_{12}) \quad\quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), *i.e.,* of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad\quad ...(8)$$

is the only suitable solution of one dimensional wave equation.

Example 18.3. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin (\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$$y(x, t) = a \sin (\pi x/l) \cos (\pi ct/l). \quad \text{(V.T.U., 2010 ; S.V.T.U., 2008 ; Kerala, 2005 ; U.P.T.U., 2004)}$$

Solution. The vibration of the string is given by $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad ...(ii) \quad\quad \text{and} \quad\quad y(l, t) = 0 \quad\quad ...(iii)$$

Since the initial transverse velocity of any point of the string is zero,

therefore, $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ ...(iv)

Also $\quad\quad y(x, 0) = a \sin (\pi x/l)$ ...(v)

Now we have to solve (i) subject to the *boundary conditions* (ii) and (iii) and *initial conditions* (iv) and (v). Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad\quad ...(vi)$$

By (ii), $\quad\quad y(0, t) = C_1(C_3 \cos cpt + C_4 \sin cpt) = 0$

For this to be true for all time, $C_1 = 0$.

Hence $\quad\quad y(x, t) = C_2 \sin px(C_3 \cos cpt + C_4 \sin cpt)$ ...(vii)

and $\quad\quad \dfrac{\partial y}{\partial t} = C_2 \sin px \, \{C_3(-cp \cdot \sin cpt) + C_4(cp \cdot \cos cpt)\}$

$\therefore$ By (iv), $\quad\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \cdot (C_4 \, cp) = 0$, whence $C_2 C_4 cp = 0$.

If $C_2 = 0$, (vii) will lead to the trivial solution $y(x, t) = 0$,

$\therefore$ the only possibility is that $C_4 = 0$.

Thus (vii) becomes $y(x, t) = C_2 C_3 \sin px \cos cpt$ ...(viii)

∴ By (iii), $y(l, t) = C_2 C_3 \sin pl \cos cpt = 0$ for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. ∴ $pl = n\pi$, i.e., $p = n\pi/l$, where n is an integer.

Hence (i) reduces to $y(x, t) = C_2 C_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$.

[These are the solutions of (i) satisfying the boundary conditions. These functions are called the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called its **spectrum**.]

Finally, imposing the last condition (v), we have $y(x, 0) = C_2 C_3 \sin \dfrac{n\pi x}{l} = a \sin \dfrac{\pi x}{l}$

which will be satisfied by taking $C_2 C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l}$ ...(ix)

Example 18.4. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 (\pi x / l)$. If it is released from rest from this position, find the displacement $y(x, t)$.*

(Rajasthan, 2006 ; V.T.U., 2003 ; J.N.T.U., 2002)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = y_0 \sin^3 \left(\dfrac{\pi x}{l} \right)$ ...(iii)

and $\left(\dfrac{\partial y}{\partial t} \right)_{t=0} = 0$ ...(iv)

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

By (ii), $y(0, t) = c_1 (c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

∴ $y(x, t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)$

Also by (ii), $y(l, t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $y(x, t) = c_2 \sin \dfrac{n\pi x}{l} \left(c_3 \cos \dfrac{cn\pi t}{l} + c_4 \sin \dfrac{cn\pi t}{l} \right)$ ...(v)

$\dfrac{\partial y}{\partial t} = \left(c_2 \sin \dfrac{n\pi x}{l} \right) \dfrac{cn\pi}{l} \left(-c_3 \sin \dfrac{cn\pi t}{l} + c_4 \cos \dfrac{cn\pi t}{l} \right)$

By (iv), $\left(\dfrac{\partial y}{\partial t} \right)_{t=0} = \left(c_2 \sin \dfrac{n\pi x}{l} \right) \dfrac{cn\pi}{l} \cdot c_4 = 0$, i.e. $c_4 = 0$.

Thus (v) becomes $y(x, t) = c_2 c_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l} = b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$

Adding all such solutions the general solution of (i) is

$y(x, t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$ ...(vi)

$\therefore$ from (iii), $\qquad y_0 \sin^3 \dfrac{\pi x}{l} = \sum\limits_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l}$

or $\qquad y_0 \left\{ \dfrac{3 \sin \dfrac{\pi x}{l} - \sin \dfrac{3\pi x}{l}}{4} \right\} = b_1 \sin \dfrac{\pi x}{l} + b_2 \sin \dfrac{2\pi x}{l} + b_3 \sin \dfrac{3\pi x}{l} + \dots$

Comparing both sides, we have

$$b_1 = 3 y_0/4, \; b_2 = 0, \; b_3 = -y_0/4, \; b_4 = b_5 = \dots = 0.$$

Hence from (vi), the desired solution is

$$y(x, t) = \dfrac{3 y_0}{4} \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l} - \dfrac{y_0}{4} \sin \dfrac{3\pi x}{l} \cos \dfrac{3\pi ct}{l}.$$

Example 18.5. *A tightly stretched flexible string has its ends fixed at x = 0 and x = l. At time t = 0, the string is given a shape defined by F(x) = μx(l − x), where μ is a constant, and then released. Find the displacement of any point x of the string at any time t > 0.*

(Bhopal, 2008 ; Madras, 2006 ; J.N.T.U., 2005 ; P.T.U., 2005)

Solution. The equation of the string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ \qquad ...(i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ \qquad ...(ii)

Also the initial conditions are $y(x, 0) = \mu x(l - x)$ \qquad ...(iii)

and $\qquad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ \qquad ...(iv)

The solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $\qquad y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore \qquad y(x, t) = c_2 \sin px \, (c_3 \cos cpt + c_4 \sin cpt)$

Also by (ii) $\qquad y(l, t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $\qquad y(x, t) = c_2 \sin \dfrac{n\pi x}{l} \left(c_3 \cos \dfrac{n\pi ct}{l} + c_4 \sin \dfrac{n\pi ct}{l} \right)$ \qquad ...(v)

$$\dfrac{\partial y}{\partial t} = \left(c_2 \sin \dfrac{n\pi x}{l} \right) \dfrac{n\pi c}{l} \left(-c_3 \sin \dfrac{n\pi ct}{l} + c_4 \cos \dfrac{n\pi ct}{l} \right)$$

$\therefore$ by (iv) $\qquad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \dfrac{n\pi x}{l} \right) \dfrac{n\pi c}{l} \cdot c_4 = 0$

Thus (v) becomes $\qquad y(x, t) = c_2 c_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l} = b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$$ \qquad ...(vi)

From (iii), $\qquad \mu(lx - x^2) = y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l}$

where $\qquad b_n = \dfrac{2}{l} \int_0^l \mu(lx - x^2) \sin \dfrac{n\pi x}{l} \, dx$, by Fourier half-range sine series

$$= \dfrac{2\mu}{l} \left\{ \left| (lx - x^2) \left(-\dfrac{\cos n\pi x/l}{n\pi/l} \right) \right|_0^l - \int_0^l (l - 2x) \left(-\dfrac{\cos n\pi x/l}{n\pi/l} \right) dx \right\}$$

$$= \frac{2\mu}{l} \cdot \frac{1}{n\pi} \left\{ \int_0^l (l - 2x) \frac{\cos n\pi x}{l} \, dx \right\} = \frac{2\mu}{n\pi} \left\{ \left| (l - 2x) \frac{\sin n\pi x/l}{n\pi/l} \right|_0^l - \int_0^l (-2) \frac{\sin n\pi x/l}{n\pi/l} \, dx \right\}$$

$$= \frac{2\mu}{n\pi} \cdot \frac{2l}{n\pi} \int_0^l \sin \frac{n\pi x}{l} \, dx = \frac{4\mu l}{n^2 \pi^2} \left| \frac{-\cos n\pi x/l}{n\pi/l} \right|_0^l = \frac{4\mu l^2}{n^3 \pi^3} \{1 - (-1)^n\}$$

Hence from (vi), the desired solution is

$$y(x, t) = \frac{4\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l}$$

$$= \frac{8\mu l^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^3} \sin \frac{(2m - 1)\pi}{l} x \cos \frac{(2m - 1)\pi c t}{l}.$$

Example 18.6. *A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x / l$. Find the displacement $y(x, t)$.*

(S.V.T.U., 2008 ; V.T.U., 2008 ; U.P.T.U., 2006)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}$$ ...(iv)

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $\qquad y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time $c_1 = 0$.

$\therefore \qquad y(x, t) = c_2 \sin px \, (c_3 \cos cpt + c_4 \sin cpt)$

Also $\qquad y(l, t) = c_2 \sin pl \, (c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $\qquad pl = n\pi \quad \text{or} \quad p = \dfrac{n\pi}{l}, n$ being an integer.

Thus $\qquad y(x, t) = c_2 \sin \dfrac{n\pi x}{l} \left(c_3 \cos \dfrac{cn\pi}{l} t + c_4 \sin \dfrac{cn\pi}{l} t \right)$

By (iii), $\qquad 0 = c_2 c_3 \sin \dfrac{n\pi x}{l} \qquad$ for all x i.e., $c_2 c_3 = 0$

$\therefore \qquad y(x, t) = b_n \sin \dfrac{n\pi x}{l} \sin \dfrac{cn\pi t}{l} \qquad$ where $b_n = c_2 c_4$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \sum b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l}$$ ...(v)

Now $\qquad \dfrac{\partial y}{\partial t} = \sum b_n \sin \dfrac{n\pi x}{l} \cdot \dfrac{cn\pi}{l} \cos \dfrac{cn\pi t}{l}$

By (iv), $\quad v_0 \sin^3 \dfrac{\pi x}{l} = \left(\dfrac{\partial y}{\partial t} \right)_{t=0} = \sum \dfrac{cn\pi}{l} b_n \sin \dfrac{n\pi x}{l}$

or $\qquad \dfrac{v_0}{4} \left(3 \sin \dfrac{\pi x}{l} - \sin \dfrac{3\pi x}{l} \right) = \sum \dfrac{cn\pi}{l} b_n \sin \dfrac{n\pi x}{l} \qquad [\because \quad \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta]$

$$= \frac{c\pi}{l} b_1 \sin \frac{\pi x}{l} + \frac{2c\pi}{l} b_2 \sin \frac{2\pi x}{l} + \frac{3c\pi}{l} b_3 \sin \frac{3\pi x}{l} + \dots$$

Equating coefficients from both sides, we get

$$\frac{3v_0}{4} = \frac{c\pi}{l}b_1, \quad 0 = \frac{2c\pi}{l}b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l}b_3, \dots$$

$$\therefore \qquad b_1 = \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_3 = \dots = 0$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi}\left(9\sin\frac{\pi x}{l}\sin\frac{c\pi t}{l} - \sin\frac{3\pi x}{l}\sin\frac{3c\pi t}{l}\right).$$

Example 18.7. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.* (Anna, 2009 ; U.P.T.U., 2002)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $\quad y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $\quad y(x, 0) = 0$ ...(iii)

and $\qquad\qquad\qquad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l - x)$ ...(iv)

As in example 18.6, the general solution of (i) satisfying the conditions (ii) and (iii) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot \sin\frac{n\pi ct}{l} \qquad \dots(v)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l} \cdot \left(\frac{n\pi c}{l}\right)$$

By (iv), $\qquad \lambda x(l - x) = \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \dfrac{\pi c}{l}\sum_{n=1}^{\infty} nb_n \sin\dfrac{n\pi x}{l}$

$\therefore \qquad \dfrac{\pi c}{l}nb_n = \dfrac{2}{l}\displaystyle\int_0^l \lambda x(l - x)\sin\dfrac{n\pi x}{l}\,dx$

$$= \frac{2\lambda}{l}\left|(lx - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}[1 - (-1)^n]$$

or $\qquad b_n = \dfrac{4\lambda l^3}{c\pi^4 n^4}[1 - (-1)^n] = \dfrac{8\lambda l^3}{c\pi^4(2m - 1)^4}$ taking $n = 2m - 1$.

Hence, from (v), the desired solution is

$$y = \frac{8\lambda l^3}{c\pi^4}\sum_{m=1}^{\infty}\frac{1}{(2m - 1)^4}\sin\frac{(2m - 1)\pi x}{l}\sin\frac{(2m - 1)\pi ct}{l}.$$

Example 18.8. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.* (Kerala, 2005)

Solution. Let B and C be the points of the trisection of the string $OA(= l)$ (Fig. 18.2). Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$ (say).

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \qquad \dots(i)$$

and the boundary conditions are

$$y(0, t) = 0 \qquad \qquad \dots(ii)$$
$$y(l, t) = 0 \qquad \qquad \dots(iii)$$
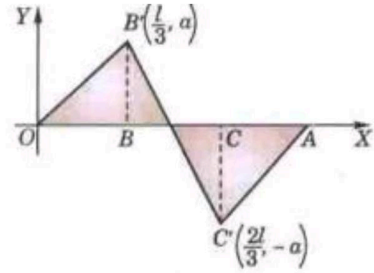$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \qquad \dots(iv)$$

Fig. 18.2

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)\, x$;

the equation of $B'C'$ is $\qquad y - a = \dfrac{-2a}{(l/3)}\left(x - \dfrac{l}{3}\right)$, i.e., $y = \dfrac{3a}{l}(l - 2x)$

and the equation of $C'A$ is $\qquad y = \dfrac{3a}{l}(x - l)$

Hence the fourth boundary condition is

$$y(x, 0) = \frac{3a}{l}\, x,\, 0 \le x \le \frac{l}{3}$$
$$= \frac{3a}{l}(l - 2x),\, \frac{l}{3} \le x \le \frac{2l}{3} \qquad \dots(v)$$
$$= \frac{3a}{l}(x - l),\, \frac{2l}{3} \le x \le l$$

As in example 18.6, the solution of (i) satisfying the boundary conditions (ii), (iii) and (iv), is

$$y(x, t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \qquad \text{[Where } b_n = C_2 C_3]$$

Adding all such solutions, the most general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \qquad \dots(vi)$$

Putting $t = 0$, we have $y(x, 0) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \qquad \dots(vii)$

In order that the condition (v) may be satisfied, (v) and (vii) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

$\therefore$ by (1) of § 10.7,

$$b_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l}\sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l - 2x)\sin \frac{n\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3a}{l}(x - l)\sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{6a}{l^2}\left[\left[x\left\{-\frac{\cos (n\pi x/l)}{(n\pi/l)}\right\} - 1\left\{-\frac{\sin (n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^{l/3}\right.$$

$$+ \left[(l - 2x)\left\{-\frac{\cos (n\pi x/l)}{(n\pi/l)}\right\} - (-2)\left\{\frac{\sin (n\pi x/l)}{(n\pi/l)^2}\right\}\right]_{l/3}^{2l/3}$$

$$\left.+ \left[(x - l)\left\{-\frac{\cos (n\pi x/l)}{(n\pi/l)}\right\} - (1) \cdot \left\{-\frac{\sin (n\pi x/l)}{(n\pi/l)^2}\right\}\right]_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos \frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin \frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos \frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin \frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos \frac{n\pi}{3}\right.$$

$$\left.+ \frac{2l^2}{n^2\pi^2}\sin \frac{n\pi}{3} - \left(\frac{l^2}{3n\pi}\cos \frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin \frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2} \cdot \frac{3l^2}{n^2\pi^2} \left(\sin \frac{n\pi}{3} - \sin \frac{2n\pi}{3} \right)$$

$$= \frac{18a}{n^2\pi^2} \sin \frac{n\pi}{3} [1 + (-1)^n] \qquad \left[\because \ \sin \frac{2n\pi}{3} = \sin \left(n\pi - \frac{n\pi}{3} \right) = -(-1)^n \sin \frac{n\pi}{3} \right]$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2} \sin \frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (*vi*) gives

$$y(x, t) = \sum_{n=2,4,\ldots}^{\infty} \frac{36a}{n^2\pi^2} \sin \frac{n\pi}{3} \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l} \qquad \text{[Take } n = 2m\text{]}$$

$$= \frac{9a}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \frac{2m\pi}{3} \sin \frac{2m\pi x}{l} \cos \frac{2m\pi c t}{l} \qquad \text{...(vii)}$$

Putting $x = l/2$ in (*vii*), we find that the displacement of the mid-point of the string, *i.e.* $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m.

This shows that the mid-point of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v.

Then $\dfrac{\partial y}{\partial x} = \dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v}$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) = \frac{\partial}{\partial u} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) + \frac{\partial}{\partial v} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) = \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \left(\frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right)$$

Substituting in (1), we get $\dfrac{\partial^2 y}{\partial u \partial v} = 0$ \qquad ...(2)

Integrating (2) w.r.t. v, we get $\dfrac{\partial y}{\partial u} = f(u)$ \qquad ...(3)

where $f(u)$ is an arbitrary function of u. Now integrating (3) w.r.t. u, we obtain

$$y = \int f(u)\, du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v. Since the integral is a function of u alone, we may denote it by $\phi(u)$. Thus

$$y = \phi(u) + \psi(v)$$

i.e.

$$y(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \text{...(4)}$$

This is the *general solution of the wave equation* (1).

Now to determine ϕ and ψ, suppose initially $u(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$.

Differentiating (4) w.r.t. t, we get $\dfrac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$

At $t = 0$, \qquad $\phi'(x) = \psi'(x)$ \qquad ...(5)

and \qquad $y(x, 0) = \phi(x) + \psi(x) = f(x)$ \qquad ...(6)

(5) gives, \qquad $\phi(x) = \psi(x) + k$

$\therefore$ \quad (6) becomes $2\psi(x) + k = f(x)$

or

$$\psi(x) = \frac{1}{2} [f(x) - k] \text{ and } \phi(x) = \frac{1}{2} [f(x) + k]$$

Hence the solution of (4) takes the form

$$y(x, t) = \frac{1}{2} [f(x + ct) + k] + \frac{1}{2} [f(x - ct) - k] = f(x + ct) + f(x - ct) \qquad \text{...(7)}$$

which is the *d'Alembert's solution* of the wave equation* (1) (V.T.U., 2011 S)

Example 18.9. *Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.* (V.T.U., 2011)

Solution. By d'Alembert's method, the solution is

$$y(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)]$$

$$= \frac{1}{2} [k\{\sin (x + ct) - \sin 2(x + ct)\} + k\{\sin (x - ct) - \sin 2(x - ct)\}]$$

$$= k[\sin x \cos ct - \sin 2x \cos 2ct]$$

Also $y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\partial y(x, 0)/\partial t = k (- c \sin x \sin ct + 2c \sin 2x \sin 2ct)_{t = 0} = 0$

i.e., the given boundary conditions are satisfied.

18.5 (1) ONE-DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section $\alpha (\text{cm}^2)$. Suppose that the sides are covered with a material impervious to heat so that the stream lines of heat-flow are all parallel and perpendicular to the area α. Take one end of the bar as the origin and the direction of flow as the positive x-axis (Fig. 18.3). Let ρ be the density (gr/cm³), s the specific heat (cal./gr. deg.) and k the thermal conductivity (cal./cm. deg. sec.).

Let $u(x, - t)$ be the temperature at a distance x from O. If δu be the temperature change in a slab of thickness δx of the bar, then by § 12.7 (ii) p. 466, the quantity of heat in this slab $= s\rho\alpha \, \delta x \delta u$. Hence the rate of increase of heat in this slab, *i.e.*, $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = R_1 - R_2$, where R_1 and R_2 are respectively the rate (cal./sec.) of inflow and outflow of heat.
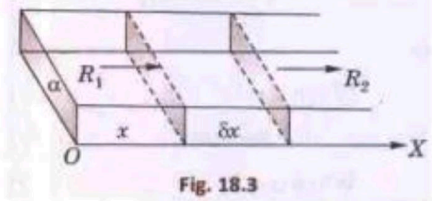

Fig. 18.3

Now by (A) of p. 466, $R_1 = - k\alpha \left(\dfrac{\partial u}{\partial x}\right)_x$ and $R_2 = - k\alpha \left(\dfrac{\partial u}{\partial x}\right)_{x + \delta x}$

the negative sign appearing as a result of (i) on p. 466.

Hence $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = - k\alpha \left(\dfrac{\partial u}{\partial x}\right)_x + k\alpha \left(\dfrac{\partial u}{\partial x}\right)_{x + \delta x}$ *i.e.*, $\dfrac{\partial u}{\partial t} = \dfrac{k}{s\rho} \left\{ \dfrac{(\partial u/\partial x)_{x + \delta x} - (\partial u/\partial x)_x}{\delta x} \right\}$

Writing $k/s\rho = c^2$, called the *diffusivity* of the substance (cm²/sec.), and taking the limit as $\delta x \to 0$, we get

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

This is the *one-dimensional heat-flow equation.* (V.T.U., 2011)

(2) Solution of the heat equation. Assume that a solution of (1) is of the form

$$u(x, t) = X(x) . T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT' = c^2 X''T, \text{ i.e., } X''/X = T'/c^2 T \qquad \text{...(2)}$$

Clearly the left side of (2) is a function of x only and the right side is a function of t alone. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2 X}{dx^2} - kX = 0 \quad \text{...(3)} \qquad \text{and} \qquad \frac{dT}{dt} - kc^2 T = 0 \qquad \text{...(4)}$$

Solving (3) and (4), we get

(i) When k is positive and $= p^2$, say :

$$X = c_1 e^{px} + c_2 e^{-px}, T = c_3 e^{c^2 p^2 t} ;$$

(ii) When k is negative and $= -p^2$, say :

$$X = c_4 \cos px + c_5 \sin px, T = c_6 e^{-c^2 p^2 t} ;$$

(iii) When k is zero :

$$X = c_7 x + c_8, T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2 p^2 t} \qquad \qquad ...(5)$$

$$u = (c_4 \cos px + c_5 \sin px) c_6 e^{-c^2 p^2 t} \qquad \qquad ...(6)$$

$$u = (c_7 x + c_8) c_9 \qquad \qquad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.*, u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.*, of the form

$$u = (C_1 \cos px + C_2 \sin px) e^{-c^2 p^2 t} \qquad \qquad ...(8)$$

is the only suitable solution of the heat equation.

Example 18.10. *Solve the equation* $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ *with boundary conditions* $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ *and* $u(1, t) = 0$, *where* $0 < x < 1, t > 0$.

Solution. The solution of the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ $\qquad \qquad ...(i)$

is $\qquad \qquad u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-p^2 t} \qquad \qquad ...(ii)$

When $x = 0$, $\qquad u(0, t) = c_1 e^{-p^2 t} = 0$ *i.e.*, $c_1 = 0$.

$\therefore$ (ii) becomes $\qquad u(x, t) = c_2 \sin px e^{-p^2 t} \qquad \qquad ...(iii)$

When $x = 1$, $\qquad u(1, t) = c_2 \sin p \cdot e^{-p^2 t} = 0$ or $\sin p = 0$

i.e., $\qquad \qquad p = n\pi.$

$\therefore$ (iii) reduces to $\qquad u(x, t) = b_n e^{-(n\pi)^2 t} \sin n\pi x$ where $b_n = c_2$

Thus the general solution of (i) is $u(x, t) = \sum b_n e^{-n^2 \pi^2 t} \sin n\pi x \qquad \qquad ...(iv)$

When $t = 0$, $3 \sin n\pi x = u(0, t) = \displaystyle\sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t} \sin n\pi x$

Comparing both sides, $b_n = 3$

Hence from (iv), the desired solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \sin n\pi x.$$

Example 18.11. *Solve the differential equation* $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

(i) *u is not infinite for* $t \to \infty$, (ii) $\dfrac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$,

(ii) $u = lx - x^2$ *for* $t = 0$, *between* $x = 0$ *and* $x = l$. $\qquad \qquad$ (P.T.U., 2007)

Solution. Substituting $u = X(x)T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad i.e., \quad X''/X = \frac{T'}{\alpha^2 T} = -k^2 \text{ (say)}$$

$$\therefore \quad \frac{d^2 X}{dx^2} + k^2 X = 0 \text{ and } \frac{dT}{dt} + k^2 \alpha^2 T = 0 \qquad \qquad \qquad ...(1)$$

Their solutions are $\quad X = c_1 \cos kx + c_2 \sin kx, \; T = c_3 e^{-k^2 \alpha^2 t} \qquad ...(2)$

If k^2 is changed to $-k^2$, the solutions are

$$X = c_4 e^{kx} + c_5 e^{-kx}, \; T = c_6 e^{k^2 \alpha^2 t} \qquad \qquad \qquad ...(3)$$

If $k^2 = 0$, the solutions are $\quad X = c_7 x + c_8, \; T = c_9 \qquad \qquad ...(4)$

In (3), $T \to \infty$ for $t \to \infty$ therefore, u also $\to \infty$ $i.e.$, the given condition (i) is not satisfied. So we reject the solutions (3) while (2) and (4), satisfy this condition.

Applying the condition (ii) to (4), we get $c_7 = 0$.

$$\therefore \quad u = XT = c_8 c_9 = a_0 \quad \text{(say)} \qquad \qquad \qquad ...(5)$$

From (2), $\quad \dfrac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx) k c_3 e^{-k^2 \alpha^2 t}$

Applying the condition (ii), we get $c_2 = 0$ and $-c_1 \sin kl + c_2 \cos kl = 0$

i.e., $\quad c_2 = 0$ and $kl = n\pi$ (n an integer)

$$\therefore \quad u = c_1 \cos kx \cdot c_3 e^{-k^2 \alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right) \frac{e^{-n^2 \pi^2 \alpha^2 t}}{l^2} \qquad ...(6)$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \Sigma a_n \cos(n\pi x/l) e^{-n^2 \pi^2 \alpha^2 t/l^2} \qquad \qquad ...(7)$$

Now using the condition (iii), we get

$$lx - x^2 = a_0 + \Sigma a_n \cos(n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l} \int_0^l (lx - x^2) \, dx = \frac{1}{l} \left| \frac{lx^2}{2} - \frac{x^3}{3} \right|_0^l = \frac{l^2}{6}$$

and $\quad a_n = \dfrac{2}{l} \displaystyle\int_0^l (lx - x^2) \cos \dfrac{n\pi x}{l} \, dx = \dfrac{2}{l} \left| (lx - x^2)\left(\dfrac{l}{n\pi} \sin \dfrac{n\pi x}{l} \right) \right.$

$$\left. - (l - 2x)\left(-\frac{l^2}{n^2 \pi^2} \cos \frac{n\pi x}{l} \right) + (-2)\left(-\frac{l^3}{n^3 \pi^3} \sin \frac{n\pi x}{l} \right) \right|_0^l$$

$$= \frac{2}{l} \left\{ 0 - \frac{l^3}{n^2 \pi^2} (\cos n\pi + 1) + 0 \right\} = -\frac{4l^2}{n^2 \pi^2} \text{ when } n \text{ is even, otherwise } 0.$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \cos\left(\frac{2m\pi x}{l} \right) e^{-4m^2 \pi^2 \alpha^2 t/l^2}.$$

Example 18.12. (a) *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.* (U.P.T.U., 2005)

(b) *Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.* (Madras, 2000 S)

Solution. (a) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \qquad \text{...(i)}$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (*steady state condition*). When u depends only on x, (i) reduces to $\partial^2 u/\partial x^2 = 0$.

Its general solution is $u = ax + b$ ...(ii)

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (ii) gives $b = 0$ and $a = 100/l$.

Thus the *initial condition* is expressed by $u(x, 0) = \dfrac{100}{l} x$ ...(iii)

Also the *boundary conditions* for the subsequent flow are

        $u(0, t) = 0$ for all values of t ...(iv)

and $u(l, t) = 0$ for all values of t ...(v)

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (i) subject to the initial condition (iii) and the boundary conditions (iv) and (v).

Now the solution of (i) is of the form

$$u(x, t) = (C_1 \cos px + C_2 \sin px) e^{-c^2 p^2 t} \qquad \qquad \text{...(vi)}$$

By (iv), $u(0, t) = C_1 e^{-c^2 p^2 t} = 0$, for all values of t.

Hence $C_1 = 0$ and (vi) reduces to $u(x, t) = C_2 \sin px \cdot e^{-c^2 p^2 t}$ ...(vii)

Applying (v), (vii) gives $u(l, t) = C_2 \sin pl \cdot e^{-c^2 p^2 t} = 0$, for all values of t.

This requires $\sin pl = 0$ *i.e.*, $pl = n\pi$ as $C_2 \neq 0$. $\therefore$ $p = n\pi/l$, where n is any integer.

Hence (vii) reduces to $u(x, t) = b_n \sin \dfrac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t / l^2}$, where $b_n = C_2$.

[These are the solutions of (i) satisfying the boundary conditions (iv) and (v). These are the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$, of the problem.]

Adding all such solutions, the most general solution of (i) satisfying the boundary conditions (iv) and (v) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t / l^2} \qquad \qquad \text{...(viii)}$$

Putting $t = 0$, $u(x, 0) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ ...(ix)

In order that the condition (iii) may be satisfied, (iii) and (ix) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l} \int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \left[x \left\{ -\frac{\cos(n\pi x/l)}{(n\pi/l)} \right\} - (1) \left\{ -\frac{\sin(n\pi x/l)}{(n\pi/l)^2} \right\} \right]_0^l = \frac{200}{l^2} \left(-\frac{l^2}{n\pi} \cos n\pi \right) = \frac{200}{n\pi} (-1)^{n+1}$$

Hence (viii) gives $u(x, t) = \dfrac{200}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{-(cn\pi/l)^2 t}$

(b) Here the initial condition remains the same as (iii) above, and the boundary conditions are

        $u(0, t) = 20$ for all values of t ...(x)

        $u(l, t) = 80$ for all values of t ...(xi)

In part (a), the boundary values (*i.e.*, the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad \qquad \text{...(xii)}$$

where $u_s(x)$ is a solution of (i) involving x only and satisfying the boundary conditions (x) and (xi) ; $u_t(x, t)$ is then a function defined by (xii). Thus $u_s(x)$ is a steady state solution of the form (ii) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t.

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (ii) we get

$$u_s(x) = 20 + (60/l)x \qquad \qquad \ldots(xiii)$$

Putting $x = 0$ in (xii), we have by (x),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \qquad \qquad \ldots(xiv)$$

Putting $x = l$ in (xii), we have by (xi),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \qquad \qquad \ldots(xv)$$

Also $\qquad \qquad u_t(x, 0) = u(x, 0) - u_s(x) = \dfrac{100x}{l} - \left(\dfrac{60x}{l} + 20\right)$ [by (iii) and $(xiii)$

$$= \dfrac{40x}{l} - 20 \qquad \qquad \ldots(xvi)$$

Hence (xiv) and (xv) give the boundary conditions and (xvi) gives the initial condition relative to the transient solution. Since the boundary values given by (xiv) and (xv) are both zero, therefore, as in part (a), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2 p^2 t}$

By (xiv), $\qquad \qquad u_t(0, t) = C_1 e^{-c^2 p^2 t} = 0$, for all values of t.

Hence $C_1 = 0$ and $\qquad u_t(x, t) = C_2 \sin px \cdot e^{-c^2 p^2 t} \qquad \qquad \ldots(xvii)$

Applying (xv), it gives $\quad u_t(l, t) = C_2 \sin ple^{-c^2 p^2 t} = 0$ for all values of t.

This requires $\qquad \qquad \sin pl = 0$, $i.e.$ $pl = n\pi$ as $C_2 \neq 0$. $p = n\pi/l$, when n is any integer.

Hence $(xvii)$ reduces to $u_t(x, t) = b_n \sin \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2}$ where $b_n = C_2$.

Adding all such solutions, the most general solution of $(xvii)$ satisfying the boundary conditions (xiv) and (xv) is

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2} \qquad \qquad \ldots(xviii)$$

Putting $t = 0$, we have $\quad u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l} \qquad \qquad \ldots(xix)$

In order that the condition (xvi) may be satisfied, (xvi) and (xix) must be same. This requires the expansion of $(40/l)\,x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\dfrac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l} \qquad \text{where} \quad b_n = \dfrac{2}{l} \int_0^l \left(\dfrac{40x}{l} - 20\right) \sin \dfrac{n\pi x}{l}\, dx = -\dfrac{40}{nx}(1 + \cos n\pi)$$

$i.e.$, $b_n = 0$, when n is odd ; $= -80/n\pi$, when n is even

Hence $(xviii)$ becomes $u_t(x, t) = \sum_{n=2, 4\ldots}^{\infty} \left(\dfrac{-80}{n\pi}\right) \sin \dfrac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t / l^2}$ [Take $n = 2m$]

$$= -\dfrac{40}{\pi} \sum_{m=1}^{\infty} \dfrac{1}{m} \sin \dfrac{2m\pi x}{l} \cdot e^{-4c^2 m^2 \pi^2 t / l^2} \qquad \qquad \ldots(xx)$$

Finally combining $(xiii)$ and (xx), the required solution is

$$u(x, t) = \dfrac{40x}{l} + 20 - \dfrac{40}{\pi} \sum_{m=1}^{\infty} \dfrac{1}{m} \sin \dfrac{2m\pi x}{l} \cdot e^{-4c^2 m^2 \pi^2 t / l^2}.$$

Example 18.13. *The ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C until steady-state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.*

Solution. Let the heat equation be $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \qquad \qquad \ldots(i)$

In steady state condition, u is independent of time and depends on x only, (i) reduces to

$$\partial^2 u / \partial x^2 = 0. \qquad \qquad \ldots(ii)$$

Its solution is $u = a + bx$

Since $u = 30$ for $x = 0$ and $u = 80$ for $x = 20$, therefore $a = 30$, $b = (80 - 30)/20 = 5/2$

Thus the initial conditions are expressed by

$$u(x, 0) = 30 + \frac{5}{2}x \qquad ...(iii)$$

The boundary conditions are $u(0, t) = 40$, $u(20, t) = 60$

Using (ii), the steady state temperature is

$$u(x, 0) = 40 + \frac{60 - 40}{20}x = 40 + x \qquad ...(iv)$$

To find the temperature u in the intermediate period,

$$u(x, t) = u_s(x) + u_t(x, t)$$

where $u_s(x)$ is the steady state temperature distribution of the form (iv) and $u_t(x, t)$ is the transient temperature distribution which decreases to zero as t increases.

Since $u_t(x, t)$ satisfies one dimensional heat equation

$$\therefore \qquad u(x, t) = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t} \qquad ...(v)$$

$$u(0, t) = 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2 t} \quad \text{whence } a_n = 0.$$

$\therefore$ (v) reduces to

$$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin px e^{-p^2 t} \qquad ...(vi)$$

Also

$$u(20, t) = 60 = 40 + 20 + \sum_{n=1}^{\infty} b_n \sin 20 pe^{-p^2 t}$$

or

$$\sum_{n=1}^{\infty} b_n \sin 20 pe^{-p^2 t} = 0 \ i.e., \ \sin 20p = 0 \ i.e., \ p = n\pi/20$$

Thus (vi) becomes

$$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-n\pi t/20} \qquad ...(vii)$$

Using (iii),

$$30 + \frac{5}{2}x = u(0, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

or

$$\frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

where $b_n = \frac{2}{20} \int_0^{20} \left(\frac{3x}{2} - 10\right) \sin \frac{n\pi x}{20} dx = -\frac{20}{n\pi}(1 + 2\cos n\pi)$

Hence from (vii), the desired solution is

$$u = 40 + x - \frac{20}{\pi} \sum \frac{1 + 2\cos n\pi}{n} \sin \frac{n\pi x}{20} e^{-(n\pi/20)^2 t}.$$

Example 18.14. Bar with insulated ends. *A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution. The temperature $u(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l (= 100 \text{ cm})$ of the bar are insulated (Fig. 18.4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0 \text{ for all } t \qquad ...(ii)$$

Initially, under steady state conditions, $\dfrac{\partial^2 u}{\partial x^2} = 0$. Its solution is $u = ax + b$.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ $\therefore$ $b = 0$ and $a = 1$.
Thus the initial condition is $u(x, 0) = x$ $0 < x < l$. ...(iii)

Now the solution of (i) is of the form $u(x, t) = (c_1 \cos px + c_2 \sin px)\, e^{-c^2 p^2 t}$ ...(iv)
Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px)\, e^{-c^2 p^2 t}$$ ...(v)

Putting $x = 0$, $\quad \left(\dfrac{\partial u}{\partial x}\right)_0 = c_2 p e^{-c^2 p^2 t} = 0$ for all t. [By (ii)]

$\therefore \qquad\qquad\qquad c_2 = 0$

Putting $x = l$ in (v), $\quad \left(\dfrac{\partial u}{\partial x}\right)_l = -c_1 p \sin pl\, e^{-c^2 p^2 t}$ for all t. [By (ii)]

$\therefore \quad c_1 p \sin pl = 0$ i.e., p being $\neq 0$, either $c_1 = 0$ or $\sin pl = 0$.
When $c_1 = 0$, (iv) gives $u(x, t) = 0$ which is a trivial solution, therefore $\sin pl = 0$.
or $\qquad\qquad pl = n\pi \qquad$ or $\qquad p = n\pi/l, \qquad n = 0, 1, 2, \dots$

Hence (iv) becomes $u(x, t) = c_1 \cos \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$.

$\therefore$ the most general solution of (i) satisfying the boundary conditions (ii) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$$ (where $A_n = c_1$) ...(vi)

Putting $t = 0$, $u(x, 0) = A_0 + \displaystyle\sum_{n=1}^{\infty} A_n \cos \dfrac{n\pi x}{l} = x$ [by (iii)]

This requires the expansion of x into a half range cosine series in $(0, l)$.

Thus $\qquad\qquad x = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos n\pi x/l \qquad\qquad$ where $a_0 = \dfrac{2}{l}\displaystyle\int_0^l x\, dx = l$

and $\qquad\qquad a_n = \dfrac{2}{l}\displaystyle\int_0^l x \cos \dfrac{n\pi x}{l}\, dx = \dfrac{2l}{n^2 \pi^2}(\cos n\pi - 1)$

$\qquad\qquad\qquad = 0$, where n is even ; $= -4l/n^2\pi^2$, when n is odd.

$\therefore \qquad\qquad A_0 = \dfrac{a_0}{2} = l/2$, and $A_n = a_n = 0$ for n even ; $= -4l/n^2\pi^2$ for n odd.

Hence (vi) takes the form

$$u(x, t) = \frac{l}{2} + \sum_{n=1,3,\dots}^{\infty} \frac{4l}{n^2\pi^2} \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-c^2 (2n-1)^2 \pi^2 t/l^2}$$ ...(vii)

This is the required temperature at a point P_1 distant x from end A at any time t.

Fig. 18.4

Obs. The sum of the temperatures at any two points equidistant from the centre is always $100°C$, a constant.

Let P_1, P_2 be two points equidistant from the centre C of the bar so that $CP_1 = CP_2$ (Fig. 18.4).
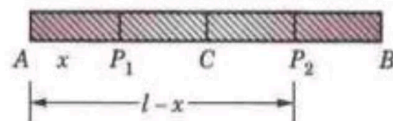If $\qquad\qquad AP_1 = BP_2 = x$ (say), then $AP_2 = l - x$.

∴ Replacing x by $l - x$ in (vii), we get the temperature at P_2 as

$$u(l - x, t) = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n - 1)^2} \cos \frac{(2n - 1)\pi(l - x)}{l} e^{\frac{-c^2(2n - 1)^2 \pi^2 t}{l^2}}$$

$$= \frac{l}{2} + \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n - 1)^2} \cos \frac{(2n - 1)\pi x}{l} e^{\frac{-c^2(2n - 1)^2 \pi^2 t}{l^2}} \qquad \text{...(viii)}$$

$$\left\{ \because \cos \frac{(2n - 1)\pi(l - x)}{l} = \cos \left[2n\pi - \pi - \frac{(2n - 1)\pi x}{l} \right] = -\cos \frac{(2n - 1)\pi x}{l} \right.$$

Adding (vii) and (viii), we get $u(x, t) + u(l - x, t) = l = 100°C$.

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